Newton's Method for PDE II

Ryan McPeck

Northern Arizona University

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General Information

- A partial differential equation, or PDE, is a differential equation involving partial derivatives.
- PDEs have many applications:
  - Harvesting
  - Starburst
  - Heat
  - Waves
- Specifically we studied $\Delta u + su + u^3 = 0$ on the unit disk centered at the origin in $\mathbb{R}^2$ with $u = 0$ on the boundaries.

Standard Form

- $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$

Newton's Method in Higher Dimensions

- $f : \mathbb{R}^m \to \mathbb{R}^m$ an arbitrary function
- $a_{k+1} = a_k - (J_f(a_k))^{-1} f(a_k)$
  - J_f is a Jacobian of the function f, or a vector of first derivatives

Theorem

- Let $J : X \to \mathbb{R}$ (Recall that X is the function space that we are working in)
- $\nabla J(u) = 0 \Longleftrightarrow u$ is a solution to the pde
- $J(u) = \int \frac{|\nabla J(u)|^2}{2} - F(u) d\hat{x}$
  - $F(u) = \frac{su^2}{2} + \frac{u^4}{4} = \int_0^u f(s) ds$

Proof in One Direction

- Suppose $u(x)$ is a solution to $u'' + u^3 = 0$. Then $\int (u'' + u^3) v \, dx = 0$ for all v.
- $J(u) = \int \frac{|\nabla J(u)|^2}{2} - \frac{u^4}{4} \, d\hat{x}$
- $J'(u)(v) = \int u'v' - u^3 v = -\int (u'' + u^3) v = - < u'' + u^3, v >$
- By definition, $\nabla J(u) = -(u'' + u^3) = 0$.

Programming

- Language: MATLAB
- Goals:
  - Find roots of $J(u) = \int \frac{|\nabla J(u)|^2}{2} - F(u)d\hat{x}$ using Newton's Method
  - Display and interpret results

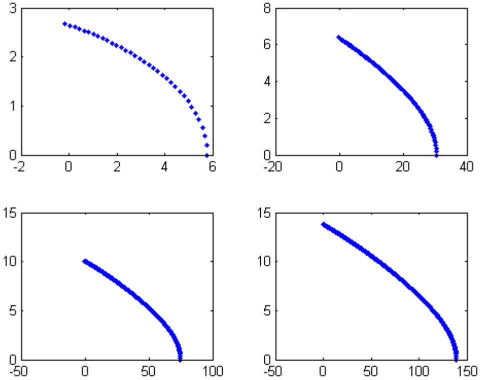
Gradient Newton Galerkin Algorithm

- Mentor and colleagues have successfully used modified Newton's Method to find solutions for PDE
- Our particular PDE of interest was studied by a student working on his master's thesis
- $a^{k+1} = a^k - (\frac{\partial g_i}{\partial a_j})^{-1} g(a^k)$
  - $\frac{\partial g_i}{\partial a_j}$ is the Hessian, a matrix filled with second derivatives
  - g is the gradient that we are trying to find roots of

Filling in the Gradient and the Hessian

- $g_i = J'(u)(\Psi_i) = a_i\lambda_i - \int f(u)\Psi_i$
- $h_{ij} = J''(u)(\Psi_i, \Psi_j) = \lambda_i\delta_{ij} - \int f'(u)\Psi_i\Psi_j$
- Integrals were computed numerically with Riemann sums
- Solve: $\dfrac{\partial g_i}{\partial a_j} \cdot \chi = g$ to avoid calculating the inverse of the Hessian
- Tolerances and iteration limits imposed to find and save off solutions

Bifurcation Diagrams



What These Solutions Look Like

- Solutions can be reassembled using linear combinations
- Sadly, we have not quite figured out the easiest way to do this yet
- Recall graphics from last presentation

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